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A THEOREM CONCERNING UNIFORM CONVERGENCE

BY GEO. D. BIRKHOFF

IN this note I aim to prove the converse of the theorem: If the series

$$(1) \qquad |u_1(x)| + |u_2(x)| + \cdots$$

converges uniformly in the interval $a \leqq x \leqq b$, then the series

$$(2) \qquad u_1(x) + u_2(x) + \cdots$$

converges uniformly in the same interval, however its terms may be re-arranged.*

The converse theorem is the following:

THEOREM: *If a series* (2) *is uniformly convergent in the interval* $a \leqq x \leqq b$ *however its terms may be re-arranged, then the absolute value series* (1) *is uniformly convergent in the same interval.*†

To show that series (1) is uniformly convergent, is to show that for every ϵ there is an N such that whenever $n \geqq N$,

$$R_n(x) < \epsilon$$

for every point x of the interval.‡

Here $R_n(x)$ denotes the remainder of the series (1) after the n^{th} term:

$$R_n(x) = |u_{n+1}(x)| + |u_{n+2}(x)| + \cdots;$$

and since all the terms of (1) are positive, $|R_n(x)| = R_n(x)$.

The proof is indirect, as follows. Suppose that under the given conditions the series (1) were *not* uniformly convergent; then for some fixed ϵ

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[*] For this theorem, and for a proof that the uniform and absolute convergence of (2) in some order does not suffice to ensure its uniform convergence in every order, see Bôcher, ANNALS OF MATHEMATICS, ser. 2, vol. 4 (1902–03), pp. 159–160.

[†] The question of the validity of this converse theorem was suggested to me by Professor E. H. Moore.

[‡] This familiar definition of uniform convergence is here repeated at length because it is not always given correctly even in the best text-books. *Cf.*, for example, Picard, *Traité d'analyse*, vol. 1, 2d edition, 1901, p. 211; or Goursat, *Cours d'analyse*, vol. 1, 1902, p. 406.

there would be, for every m, some integer $n \geqq m$ and some point x_m of the interval, such that

$$R_n(x_m) > \epsilon.$$

Indeed, since $R_m(x_m) \geqq R_n(x_m)$, we may say that for every integer m there would be some point x_m such that

$$R_m(x_m) > \epsilon.$$

But since (2) is convergent for $x = x_m$, however its terms may be re-arranged, we know that the absolute value series (1) is convergent for $x = x_m$; so that we can find an integer $r > m$ such that $R_{m+r}(x_m) < R_m(x_m) - \epsilon$, whence $R_m(x_m) - R_{m+r}(x_m) > \epsilon$,
or

$$|u_{m+1}(x_m)| + |u_{m+2}(x_m)| + \cdots + |u_{m+r}(x_m)| > \epsilon. \tag{3}$$

This inequality may be used as a recurrent formula for the determination of an infinite sequence of integers,

$$m_1, m_2, m_3, \cdots,$$

as follows: when $m = 0$, $r = m_1$; then $m_1 + r_1 = m_2$; $\cdots$; $m_i + r_i = m_{i+1}$, where the r_i corresponding to each m_i is so determined that (3) holds. These integers divide both the given series into sections or blocks, the i^{th} section, for example, containing the terms from the $(m_i + 1)^{\text{th}}$ to the $(m_{i+1})^{\text{th}}$, inclusive.

Consider a typical section of series (2), namely

$$u_{m+1}(x_m) + u_{m+2}(x_m) + \cdots + u_{m+r}(x_m); \tag{4}$$

and let S'_m be the sum of the absolute values of the terms of one sign in this section, and S''_m the sum of the absolute values of the remaining terms (if any). Then from (3),

$$S'_m + S''_m > \epsilon,$$

so that one or other of the two quantities S'_m and S''_m must be $> \epsilon/2$; we may assume that

$$S'_m > \frac{\epsilon}{2}. \tag{5}$$

Now re-arrange the terms of (4) in such a way that all the terms which belong to S'_m come first; and do the same in each section of the series. The result will be a certain re-arrangement of series (2) which may be denoted by

$$(6) \qquad U_1(x) + U_2(x) + U_3(x) + \cdots.$$

We proceed to show that, contrary to hypothesis, this new series (6) cannot be uniformly convergent.

For, if the first p terms ($p \leqq r$) in a typical section of (6) are the terms which belong to S'_m, then, by (5),

$$| U_{m+1}(x_m) + U_{m+2}(x_m) + \cdots + U_{m+p}(x_m) | > \frac{\epsilon}{2} ;$$

that is, for every N there are two integers, m and $m+p$ (where $N < m < m+p$), and some point x_m, such that

$$| R'_m(x_m) - R'_{m+p}(x_m) | > \frac{\epsilon}{2} ,$$

where R' denotes the remainder in the series (6); and therefore either $|R'_m(x_m)|$ or else $|R'_{m+p}(x_m)|$ is $> \epsilon/4$.

Hence the supposition with which we started must be false, and the theorem is proved.

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